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Twistor Dynamics of a Massless Spinning Particle[1]

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The twistor Hamiltonian dynamics of a massless particle with helicity moving under the action of an external central force is formulated. The nonholomorphic canonical twistor quantization procedure turns the so-formulated Hamiltonian dynamic into its quantum analogue.

1. INTRODUCTION

The purpose of this paper is to demonstrate the utility of the twistor phase space (Penrose, 1968b, 1972; Tod, 1979; Zakrzewski, 1995; Bette, 1996; Bette and Zakrzewski, 1996, 1997). Twistors may be regarded as spinors of the $SO(4,2)$ group, which is a two-to-one covering group of the conformal group $C(3,1)$ of the compactified Minkowski-space, while $SU(2,2)$, acting on the twistor vector space, is the universal covering group of the group $SO(4,2)$. More exactly, the chain of the involved two-to-one group homomorphisms is given by $SU(2,2) \rightarrow SO(4,2) \rightarrow C(3,1)$.

As recognized by Penrose, a twistor may be used to represent phase space variables of a massless particle with, in general, nonvanishing classical limit of its helicity. Relative to an inertial frame and relative to an arbitrary three-space origin in this frame, the instantaneous physical variables of the massless particle (signal) may be identified as its translational kinetic energy E, its direction of motion $\bar{n}$, the non-covariant position $\bar{r}$ of its center of (translational + spinning) energy, the value of its helicity s, and, finally, the phase of the twistor, which is canonically conjugate to its helicity. Altogether we have eight variables which match the eight real dimensions of the twistor

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phase space. A spinning massless particle is not localized in the Minkowski space, or in other words, there is no single world-curve associated to it (Penrose and MacCallum, 1972; Odzijewicz *et al.*, 1986).

In order to show how the notion of a twistor phase space may be used, it seemed natural to us to analyze a simple (hypothetical?) example in which a massless particle with a nonvanishing helicity is moving under the action of a conservative and central force. After quantization, this simple model could be used in high-energy particle physics with the massless particle being identified with a confined quark moving in an effective approximately conservative and central force field created by the remaining parts of the elementary particle under study. Perhaps by an appropriate choice of the potential found by means of an educated guess, some of the hadronic resonances could then be recovered.

Such a classical system is completely integrable by quadratures. This is so because there exist *à priori* four mutually Poisson commuting constants of motion on the (8D) twistor phase space, namely the Hamiltonian generating the motion, the absolute value of the total angular momentum, the value of an arbitrary component of the total angular momentum, and the value of the helicity. After quantization, the corresponding operators define a maximal set of mutually commuting observables.

Twistor theory has been around for many years and has drawn the attention of a number of physicists (e.g., Ablamowicz *et al.*, 1982).

2. TWISTOR SPACE AS A RELATIVISTIC PHASE SPACE

Twistor space **T** is a $C^4 = \{(Z^0, Z^1, Z^2, Z^3)\}$ with a pseudo-Hermitian $SU(2,2)$ conformally invariant metric $\rho := Z^a \overline{W}_a$ ($\overline{W}_0 = \overline{W^2}$, $\overline{W}_1 = \overline{W^3}$, $\overline{W}_2 = \overline{W^0}$, $\overline{W}_3 = \overline{W^1}$) (Penrose, 1967, 1975; Penrose and MacCallum, 1972). Such a choice of the representation of the $SU(2,2)$ metric is motivated by the fact that while restricting $SU(2,2)$ to its Lorentz subgroup, a twistor Z splits naturally into its two Weyl spinor components,

$$Z^\alpha = (\omega^A, \pi_{A'}), \qquad \overline{Z}_\alpha = (\overline{\pi}_A, \overline{\omega}^{A'})$$

The omega spinor component of the twistor Z is a translation-dependent entity, mixing itself with the pi component. In addition, the pi component is a conformally dependent entity, mixing itself with the omega component when a special conformal transformation is applied. Consequently a twistor, which is a spinor of the group $SO(4,2)$, is *not equivalent* with two Weyl spinors, i.e., spinors of the Lorentz group $SO(3,1)$. The equivalence is sometimes wrongly claimed. However, if the origin of the Minkowski space is fixed and the infinity of the compactified Minkowski space is selected, a twistor may be *represented* by two Weyl spinors or if one so wishes by a

Dirac bispinor. See a discussion from a different point of view in Luehr and Rosenbaum (1982).

The imaginary part of the $SU(2,2)$ invariant metric ρ defines a symplectic structure Ω on **T**, i.e., a conformally and therefore Poincaré-invariant canonical Poisson bracket algebra (Penrose, 1968b, 1972; Tod, 1979),

$$\{\overline{Z}_\beta, Z^\alpha\} = i\delta_\beta^\alpha$$

with all the remaining commutation relations being equal to zero. In terms of the Weyl spinor components, the only nonvanishing commutations relations are

$$\{\overline{\pi}_B, \omega^A\} = i\delta_B^A \tag{1}$$

The 10 real-valued functions representing relativistic observables on **T** [we adopt the so-called abstract index notation (Penrose, 1968a; Penrose and Rindler, 1984) where it seems appropriate]

$$P_a := \pi_{A'}\overline{\pi}_A, \qquad M_{ab} := i\overline{\omega}_{(A'}\pi_{B')}\epsilon_{AB} + \text{c.c.} \tag{2}$$

define a momentum mapping of the Poincaré group into the twistor phase space **T**, or in other words, the commutation relations, induced by (1), among these 10 real-valued functions in (2) represent the Poincaré algebra (Hughston, 1979; Woodhouse, 1992). Using spinor algebra calculus, it can also be seen that (Penrose and MacCallum, 1972)

$$S^a := \tfrac{1}{2}\,\epsilon^{abcd}M_{bc}P_d = sP^a = s\pi^{A'}\overline{\pi}^A, \qquad \text{where} \quad s := \tfrac{1}{2}(Z^\alpha\overline{Z}_\alpha) \tag{3}$$

The real-valued, conformally invariant function s in (3), the $SU(2,2)$ norm of Z, represents the classical limit of the helicity operator (Penrose and MacCallum, 1972; Penrose, 1975).

3. CLASSICAL TWISTOR DYNAMICS OF A SPINNING MASSLESS SIGNAL

Assume that the inertial frame in which an external conservative and central force described by a potential energy $U(|\bar{r}|)$ is given. Let $\bar{r} = 0$ in this inertial frame denote a fixed space origin and identify it with the location of the source producing the force acting on the massless particle. Then $\bar{r}$ denotes the instantaneous position of the center of the (translational + spinning) energy of the massless signal relative to the so-defined fixed space origin in the considered inertial frame. The inertial frame itself may be characterized by a fixed spinor dyad,

$$\overline{\alpha}_A, \overline{\beta}_A \qquad \text{such that} \quad \overline{\alpha}^A\overline{\beta}_A = \frac{1}{\sqrt{2}} \tag{4}$$

The fixed orthonormal tetrad, a timelike base four-vector and three spacelike base four-vectors in the above-mentioned inertial frame in the Minkowski space, is then given by

$$\begin{aligned} t^a &:= \alpha^{A'}\overline{\alpha}^A + \overline{\beta}^{A'}\overline{\beta}^A, \qquad u^a_{(1)} \equiv u^a := \alpha^{A'}\overline{\alpha}^A - \beta^{A'}\overline{\beta}^A \\ u^a_{(2)} &\equiv v^a := \alpha^{A'}\overline{\beta}^A + \beta^{A'}\overline{\alpha}^A, \qquad u^a_{(3)} \equiv w^a =: i(\alpha^{A'}\overline{\beta}^A - \beta^{A'}\overline{\alpha}^A) \end{aligned} \tag{5}$$

We may call the direction defined by u^a the z-axis direction, the direction defined by v^a the x-axis direction, and the direction defined by w^a the y-axis direction.

The instantaneuos translational kinetic energy of the massless particle, the components of the instantaneuos linear three-momentum defining its direction of motion, and the functions representing the components of the total angular momentum may be regarded as functions on the twistor phase space according to

$$\begin{aligned} E := \pi_{C'}\overline{\pi}_C t^{CC'}, \qquad P_{(\alpha)} := \pi_{C'}\overline{\pi}_C u^{CC'}_{(\alpha)}, \\ \alpha = 1, 2, 3, \quad \overline{p} := (p_{(2)}, p_{(3)}, p_{(1)}) \end{aligned} \tag{6}$$

$$J_{(\alpha)} := \frac{1}{2}\,\epsilon_{(\alpha)(\gamma)(\delta)} M_{ab} u^a_{(\gamma)} u^b_{(\delta)}, \qquad \alpha = 1, 2, 3, \quad \overline{J} := (J_{(2)}, J_{(3)}, J_{(1)}) \tag{7}$$

$$\epsilon_{(\alpha)(\beta)(\gamma)} := \epsilon_{abcd} t^a u^b_{(\alpha)} u^c_{(\beta)} u^d_{(\gamma)}, \qquad \epsilon_{(1)(2)(3)} = 1$$

with ϵ_{abcd} being the completely antisymmetric Lorentz tensor defining space-time orientation. Explicitly, in terms of spinor components of the twistor variable, this yields

$$J_z = J_{(1)} = -\left[\frac{1}{\sqrt{2}}(\overline{\alpha}_A\overline{\beta}^B + \overline{\beta}_A\overline{\alpha}^B)\overline{\pi}_B\omega^A + \frac{1}{\sqrt{2}}(\alpha_{A'}\beta^{B'} + \beta_{A'}\alpha_{B'})\pi_B\overline{\omega}^{A'}\right] \tag{8}$$

$$J_x = J_{(2)} = \left[\frac{1}{\sqrt{2}}(\overline{\alpha}_A\overline{\alpha}^B - \overline{\beta}_A\overline{\beta}^B)\overline{\pi}_B\omega^A + \frac{1}{\sqrt{2}}(\alpha_{A'}\beta^{B'} - \beta_{A'}\beta^{B'})\pi_B\overline{\omega}^{A'}\right. \tag{9}$$

$$J_y = J_{(3)} = \frac{-i}{\sqrt{2}}(\overline{\alpha}_A\overline{\alpha}^B + \overline{\beta}_A\overline{\beta}^B)\overline{\pi}_B\omega^A + \frac{i}{\sqrt{2}}(\alpha_{A'}\alpha^{B'} + \beta_{A'}\alpha^{B'})\pi_{B'}\overline{\omega}^{A'} \tag{10}$$

The three functions representing the components of the position of the instantaneous center of (translational + spinning) energy are given by

$$y_{(\alpha)} := \frac{M_{ba} t^b u^a_{(\alpha)}}{P_a t^c}, \qquad \alpha = 1, 2, 3, \quad \bar{r} := (y_{(2)}, y_{(3)}, y_{(1)}) \tag{11}$$

$$z = \frac{-i}{\pi_{C'} \bar{\pi}_C t^{CC'}} \left[\frac{1}{\sqrt{2}} (\bar{\alpha}_A \bar{\beta}^B + \bar{\beta}_A \bar{\alpha}^B) \bar{\pi}_B \omega^A - \frac{1}{\sqrt{2}} (\alpha_{A'} \beta^{B'} + \beta_{A'} \alpha^{B'}) \pi_{B'} \bar{\omega}^{A'} \right] \tag{12}$$

$$x = \frac{-i}{\pi_{C'} \bar{\pi}_C t^{CC'}} \left[\frac{1}{\sqrt{2}} (\bar{\beta}_A \bar{\beta}^B - \bar{\alpha}_A \bar{\alpha}^B) \bar{\pi}_B \omega^A - \frac{1}{\sqrt{2}} (\beta_{A'} \beta^{B'} - \alpha_{A'} \alpha^{B'}) \pi_{B'} \bar{\omega}^{A'} \right] \tag{13}$$

$$y = \frac{1}{\pi_{C'} \bar{\pi}_C t^{CC'}} \left[\frac{1}{\sqrt{2}} (\bar{\alpha}_A \bar{\alpha}^B + \bar{\beta}_A \bar{\beta}^B) \bar{\pi}_B \omega^A + \frac{1}{\sqrt{2}} (\alpha_{A'} \alpha^{B'} + \beta_{A'} \beta^{B'}) \pi_{B'} \bar{\omega}^{A'} \right] \tag{14}$$

The helicity s in (3) is a conformal scalar and thereby also a Poincaré scalar function on **T**. Therefore the function s Poisson-commutes with all the functions introduced in (6)–(14). The identifications of the physical variables above are such that, using the familiar three-vector notation, one obtains that

$$\bar{J} = \bar{r} \times \bar{p} + \frac{s\bar{p}}{E} \tag{15}$$

The canonical commutation relations in (1) induce the following nonvanishing Poisson bracket relations among the above-introduced dynamical physical variables:

$$\{y_{(\alpha)}, y_{(\beta)}\} = \frac{s\epsilon_{(\alpha)(\beta)(\gamma)} p_{(\gamma)}}{E^3} \tag{16}$$

$$\{p_{(\beta)}, y_{(\alpha)}\} = \delta_{(\alpha)(\beta)}, \qquad \{E, y_{(\alpha)}\} = \frac{p(\alpha)}{E}, \qquad \{J_{(\alpha)}, J_{(\beta)}\} = \epsilon_{(\alpha)(\beta)(\gamma)} J_{(\gamma)} \tag{17}$$

$$\{J_{(\alpha)}, y_{(\beta)}\} = \epsilon_{(\alpha)(\beta)(\gamma)} y_{(\gamma)}, \qquad \{J_{(\alpha)}, p_{(\beta)}\} = \epsilon_{(\alpha)(\beta)(\gamma)} p_{(\gamma)} \tag{18}$$

The commutation relations in (16)–(18) are quite reasonable from the physical point of view. Apart from (16), they are what one should expect. Taking (16) seriously should, at the quantum level, mean that the position of the (translational + spinning) energy of a massless spinning particle, as identified in (11), is not a sharp observable.

We assume that the total energy H of a massless particle moving under the action of a conservative central potential $U(|\bar{r}|) = U(r)$ is given in the usual way:

$$H = E + U(r) \tag{19}$$

H is a real-valued function on the twistor phase space. The canonical flow induced by H on the twistor phase space may be represented by the following set of canonical equations of motion:

$$\dot{\omega}^A = \{H, \omega^A\} = i\,\frac{\partial H}{\partial \overline{\pi}_A}, \qquad \dot{\pi}_{B'} = \{H, \pi_{B'}\} = i\,\frac{\partial H}{\partial \overline{\omega}^{B'}} \tag{20}$$

For the functions identified as physical variables in (6)–(14), it yields

$$\dot{y}_{(\alpha)} = \{H, y_{(\alpha)}\} = \{E + U, y_{(\alpha)}\} = \frac{p_{(\alpha)}}{E} - \frac{s\epsilon_{(\alpha)(\beta)(\gamma)}p_{(\gamma)}}{E^3}\,\frac{\partial U}{\partial y_{(\beta)}}$$

$$\dot{p}_{(\alpha)} = \{E + U, p_{(\alpha)}\} = \{U, p_{(\alpha)}\} = \frac{\partial U}{\partial y_{(\beta)}}\,\{y_{(\beta)}, p_{(\alpha)}\} = -\frac{\partial U}{\partial y_{(\alpha)}}$$

$$\dot{J}_{(\alpha)} = 0, \qquad \dot{s} = 0 \tag{21}$$

which in the three-vector notation reads

$$\dot{\overline{r}} = \frac{\overline{p}}{E} - \frac{s}{E^3}\,(\overline{r} \times \overline{p})\,\frac{f(r)}{r^2}, \qquad \dot{\overline{p}} = \frac{f(r)}{r^2}\,\overline{r}, \tag{22}$$

$$\dot{\overline{J}} = 0, \quad \dot{s} = 0, \quad f(r) := -r\,\frac{dU}{dr}$$

From (2), it follows that $E = |\overline{p}|$, which, together with (15), implies

$$(\overline{r} \cdot \overline{p})^2 = E^2 r^2 + s^2 - J^2 \tag{23}$$

Choosing the direction of the constant total angular momentum along the positive direction of the z axis, i.e., along u^a in (5) yields

$$\overline{J} = J\overline{e}_z \qquad (u^a = \overline{e}_z) \tag{24}$$

Therefore, from (15), (19), (23), and (24) it follows that

$$z = \pm\frac{rs}{J}\,\sqrt{1 + \frac{s^2 - J^2}{(H - U(r))^2 r^2}} \tag{25}$$

Introducing the plane polar coordinates ρ and φ relative to the origin in the plane spanned by the two mutually orthogonal spacelike unit vectors v^a and w^a in (5), remembering that

$$r = |\overline{r}| = \sqrt{\rho^2 + z^2} \tag{26}$$

where ρ and z are the cylinder coordinates of the position vector $\overline{r}$ relative to the origin, and again using (15), (19), and (24), we obtain

$$p_\varphi = \frac{(H - U(r))^2\,\rho J}{s^2 + r^2(H - U(r))^2}, \qquad p_\rho = \frac{(H - U(r))^2\,Jz\rho}{s(s^2 + r^2(H - U(r))^2)} \tag{27}$$

$$p_z = \frac{(H - U(r))J(s^2 + z^2(H - U(r))^2)}{s(s^2 + r^2(H - U(r))^2)} \tag{28}$$

Finally, making use of (22), the remaining equations become

$$\dot{r} = \sqrt{1 + \frac{s^2 - J^2}{(H - U(r))^2 r^2}}, \qquad \dot{\phi} = \frac{(H - U(r))J}{s^2 + r^2(H - U(r))^2}\left(1 + \frac{s^2 f(r)}{r^2 E^2}\right) \tag{29}$$

These equations are completely integrable by quadratures. Taking, e.g., $U(r) = \frac{1}{2}kr^2$ or $U(r) = k/r$, i.e., assuming the 3D harmonic oscillator or Kepler potential, amounts to a straighforward, although nontrivial integration procedure. This will produce space trajectories swept out by the center of (translational + spinning) energy of the massless signal. In addition, the solutions describe how the the linear momentum of the massless particle, i.e., its direction of translational motion, changes with time. The explicit calculations are not easily performed by hand, but may be done using, e.g., the formidable computer program Maple V. We hope to be able to present results of such explicit calculations in forthcoming papers.

The absolute value of the velocity of the center of (translational + spinning) energy $\dot{\bar{r}}$ is larger than one (i.e., is *larger than the velocity of light*). However, the absolute value of the translational velocity of the massless particle, i.e., $|\bar{p}|/E$ is, by definition, always equal to one. However, in the limit when the helicity s (or/and the interaction) vanishes, the two velocities coincide.

4. QUANTUM TWISTOR DYNAMICS OF A SPINNING MASSLESS SIGNAL

A nonholomorphic quantization procedure corresponding to the so-called real polarization of the twistor phase space (Penrose and MacCallum, 1972; Woodhouse, 1992) is obtained by means of a natural prescription à la Dirac (Dirac, 1958) given by

$$\hat{\omega}^A := \frac{\partial}{\partial \bar{\pi}_A}, \qquad \hat{\bar{\omega}}^{A'} := -\frac{\partial}{\partial \pi_{A'}}, \qquad \hat{\bar{\pi}}_A := \bar{\pi}_A, \qquad \hat{\pi}_{A'} := \pi_{A'} \tag{30}$$

The Poisson brackets relations in (1) will hereby be replaced by the corresponding commutators turning the classical twistor phase space dynamics of a massless particle into its quantum mechanical analogue.

By the use of (30), the linear four-momentum functions, the angular four-momentum functions in (2), and the helicity function in (3) turn into the corresponding operators

$$\hat{P}_a := \bar{\pi}_A \pi_{A'}, \qquad \hat{M}^{ab} := i\bar{\pi}^{(A} \frac{\partial}{\partial \bar{\pi}_{B)}} \epsilon^{A'B'} + i\pi^{(A'} \frac{\partial}{\partial \pi_{B')}} \epsilon^{AB} \tag{31}$$

$$\hat{s} := \frac{1}{2}\left(\bar{\pi}_A \frac{\partial}{\partial \bar{\pi}_A} - \pi_{A'} \frac{\partial}{\partial \pi_{A'}}\right) \tag{32}$$

The canonical Poisson bracket relations in (1) imply that the Poisson bracket relations among the functions in (2) satisfy the Poisson commutation of the Poincaré algebra. Therefore the canonical commutation relations between the differential and multiplicative operators in (30) ensure automatically that the operators in (31) obey commutation relations of the Poincaré algebra. We remind the reader that the helicity operator $\hat{s}$ in (32) commutes with the operators in (31).

We introduce a Poincaré invariant scalar product on the space of complex-valued (nonholomorpic) functions of $\pi_{A'}$, and $\overline{\pi}_A$ in the following way:

$$\langle g_1|g_2\rangle := \int [\overline{g}_1(\overline{\pi}_B, \pi_{B'})g_2(\pi_{B'}, \overline{\pi}_B)]\, d\pi^{A'} \wedge d\pi_{A'} \wedge d\overline{\pi}^A \wedge d\overline{\pi}_A \tag{33}$$

Then the set of such functions having finite norm with respect to the scalar product in (33) defines quantum mechanical states of a massless spinning particle (signal) in a representation which may be called the square root of the linear momentum representation.

In the inertial frame as described in the previous section, one has (there is a sign ambiguity here due to the square root representation)

$$\overline{\alpha}^A\overline{\pi}_A = \pm\sqrt{E}e^{i\varphi/2}\, e^{-i\psi} \sin\frac{\theta}{2}, \qquad \alpha^{B'}\pi_{B'} = \pm\sqrt{E}e^{-i\varphi/2}\, e^{i\psi} \sin\frac{\theta}{2}$$

$$\overline{\beta}^B\overline{\pi}_B = \pm\sqrt{E}e^{-i\varphi/2}\, e^{-i\psi} \cos\frac{\theta}{2}, \qquad \beta^{A'}\pi_{A'} = \pm\sqrt{E}e^{i\varphi/2}\, e^{i\psi} \cos\frac{\theta}{2}$$

and inversely [the angle coordinate ψ represents the phase canonically conjugate to the helicity operator $\hat{s}$ in (32)],

$$E = (\alpha^{B'}\overline{\alpha}^B + \beta^{B'}\overline{\beta}^B)\pi_{B'}\overline{\pi}_B$$

$$e^{4i\psi} = \frac{(\alpha^{A'}\pi_{A'})(\beta^{B'}\pi_{B'})}{(\overline{\beta}^A\overline{\pi}_A)(\overline{\alpha}^B\overline{\pi}_B)}, \qquad e^{2i\varphi} = \frac{p_{(2)} + ip_{(3)}}{p_{(2)} - ip_{(3)}} = \frac{(\overline{\alpha}^A\overline{\pi}_A)(\beta^{A'}\pi_{A'})}{(\overline{\beta}^B\overline{\pi}_B)(\alpha^{B'}\pi_{B'})}$$

$$\cos\theta = \frac{p_{(1)}}{E} = \frac{(\beta^{A'}\overline{\beta}^A - \alpha^{A'}\overline{\alpha}^A)\pi_{A'}\overline{\pi}_A}{(\alpha^{B'}\overline{\alpha}^B + \beta^{B'}\overline{\beta}^B)\pi_{B'}\overline{\pi}_B}$$

From the above considerations, it now follows that the Hermitian differential operators corresponding to the three components of the total angular momentum $J_{(1)}$, $J_{(2)}$, $J_{(3)}$ in (8)–(10) are given by

$$\hat{J}_{(1)} = -\frac{1}{\sqrt{2}}(\overline{\alpha}_A\overline{\beta}^B + \overline{\alpha}^B\overline{\beta}_A)\overline{\pi}_B\frac{\partial}{\partial\overline{\pi}_A} + \frac{1}{\sqrt{2}}(\alpha_{A'}\beta^{B'} + \beta_{A'}\alpha^{B'})\pi_{B'}\frac{\partial}{\partial\pi_{A'}}$$

$$\hat{J}_{(2)} = -\frac{1}{\sqrt{2}}(\overline{\beta}_A\overline{\beta}^B - \overline{\alpha}_A\overline{\alpha}^B)\overline{\pi}_B\frac{\partial}{\partial\overline{\pi}_A} + \frac{1}{\sqrt{2}}(\beta_{A'}\beta^{B'} - \alpha_{A'}\alpha^{B'})\pi_{B'}\frac{\partial}{\partial\pi_{A'}}$$

$$\hat{J}_{(3)} = -\frac{i}{\sqrt{2}}(\overline{\alpha}_A\overline{\alpha}^B + \overline{\beta}_A\overline{\beta}^B)\overline{\pi}_B\frac{\partial}{\partial\overline{\pi}_A} - \frac{i}{\sqrt{2}}(\alpha_{A'}\alpha^{B'} + \beta_{A'}\beta^{B'})\pi_{B'}\frac{\partial}{\partial\pi_{\alpha'}}$$

and similarily for the components of the position (of the center of translational + spinning energy) operators corresponding to the functions in (12)–(14). We do not write these operator expressions explicitly here, but note that the square of the classical distance (between the origin and the center of translational + spinning energy) is given by $[y_{(\alpha)}P_{(\alpha)} = (i/2)(\overline{\pi}_A\omega^A - \pi_{A'}\overline{\omega}^{A'})]$

$$r^2 = y_{(\alpha)}y_{(\alpha)} = \frac{J^2 - s^2 + (y_{(\alpha)}p_{(\alpha)})^2}{E^2} = \frac{J^2 - s^2 - \frac{1}{4}(\overline{\pi}_A\omega^A - \pi_{A'}\overline{\omega}^{A'})^2}{E^2}$$

which, employing the normal ordering of terms, implies that the square of the quantum distance operator (measuring the square of the distance between the origin and the center of translational + spinning energy) is represented by the following differential operator:

$$\hat{r}^2 := \frac{1}{E^2}(\hat{J}^2 - \hat{s}^2 - \hat{\kappa}^2), \qquad \hat{\kappa} := \frac{1}{2}\left(\overline{\pi}_A\frac{\partial}{\partial\overline{\pi}_A} + \pi_{A'}\frac{\partial}{\partial\pi_{A'}} + 4\right)$$

Ignoring the usual ordering problems (for example, adopt the normal ordering), the way is now open to calculate the (energy) spectrum of any operator of the form

$$\hat{H} = E + U(\hat{r})$$

Note that $\hat{H}$, $\hat{J}^2$, $\hat{J}_{(1)}$, and $\hat{s}$ form a complete set of mutually commuting observables.

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